1. Introduction

Oscillatory dynamics can be observed in a variety of systems in the animate and inanimate world. Systems that are able to produce oscillatory dynamics by themselves are known as self-oscillators, with examples of particular interest including the Rayleigh oscillator [Rayleigh, 1945] and the van der Pol oscillator [van der Pol, 1934] (for a review see Jenkins [2013]). These systems belong to the even broader class of active systems [Ebeling et al., 1999; Mikhailov, 1990; Romanczuk et al., 2012; Schweitzer, 2003], see Fig. 1, that either feature energy depots or are in contact with pumping sources. Importantly, the pumping sources and energy depots do not specify the oscillatory dynamics. Rather, the oscillations emerge via bifurcations (e.g., Hopf bifurcations [Guckenheimer & Holmes, 1983]). On a mechanistic level, the pumping source
is frequently related to a friction (or damping) force that is velocity-dependent such that at relatively low speeds the friction force effectively becomes a pumping force [Ebeling, 2004; Lindner & Nicola, 2008; Schweitzer et al., 1998; Yushchenko, 2016; Yushchenko & Badalyan, 2012]. Self-propagation in chemical systems [Mikhailov & Meinköhn, 1997; Sumino & Yoshikawa, 2008] and biological systems [Bödeker et al., 2003; Schienbein et al., 1994; Schienbein & Gruler, 1993] shows that experimental data is consistent with the aforementioned concept of velocity-dependent friction. In the context of the SET model introduced by Schweitzer, Ebeling, and Tilch [Schweitzer et al., 1998, 2001] and the model discussed by Olemskoi, Khomenko, Yushchenko and colleagues [Khomenko & Yushchenko, 2003; Olemskoi, 2002; Olemskoi & Khomenko, 2001; Olemskoi et al., 2011; Yushchenko & Badalyan, 2012], implications of the negative friction concept have been discussed, for example, for swarming dynamics [Ebeling, 2003; Ebeling & Schimansky-Geier, 2008; Erdmann & Ebeling, 2003; Erdmann et al., 2005; Frank, 2016; Schweitzer et al., 1998, 2001], foraging dynamics [Ebeling et al., 1999, 2005; Erdmann et al., 2000; Streffer et al., 2009], evacuation situations [Czirok et al., 1999; Czirok & Vicsek, 2000], and active particles moving in ratchets [Burada & Lindner, 2012; Ebeling et al., 2008; Fiasconaro et al., 2008, 2013; Schweitzer et al., 2000]. As far as self-oscillators are concerned, general characteristics of oscillators featuring velocity-dependent friction have been discussed from the perspectives of physics [Klimontovich, 1983] and mathematics [Verhulst, 1996]. Applications to the study of human rhythmic movements [Beek et al., 1995, 1996; Eisenhammer et al., 1991; Kelso et al., 1981; Kay et al., 1987, 1991; Mottet & Bootsma, 1999; Silva et al., 2009; van Mourik et al., 2006] and properties of the human cochlea [Reit et al., 2014] can be found. In electronic engineering, electronic self-oscillators play a crucial role as elementary components of more complex circuits. In this context, pumping mechanisms are often realized by negative resistance elements. Benchmark examples of electronic self-oscillators whose dynamics can be described by means of negative resistance components are the Meissner oscillator that under certain conditions satisfies the van der Pol oscillator equation [Mathis & Bremer, 2009], the Colpitts oscillator [Hirschel, 1960], and the Walker-Connelly oscillator [Mickens, 1989]. In order to get insights into active systems in general and self-oscillators in particular, the canonical-dissipative approach [Ebeling, 2000; Ebeling & Sokolov, 2004; Feistel & Ebeling, 1989; Frank, 2002, 2005; Graham, 1973; Haken, 1973; Hongler & Ryter, 1978; Schweitzer et al., 2001; Romanczuk et al., 2012], see Fig. 1, is an especially promising approach because it often allows for semi-analytical treatment of the problem at hand. According to the canonical-dissipative approach, the Hamiltonian function that determines the system of interest in the absence of damping and pumping is used to describe the negative friction element that turns the system into an active system (e. g., a self-propagating particle or a self-oscillator). The canonical-dissipative oscillator (CDO) thus obtained [Feistel & Ebeling, 1989; Frank, 2005, 2010c; Ebeling & Sokolov, 2004] allows for an analytical description of its limit cycle attractor [Ebeling et al., 2008; Ebeling & Sokolov, 2004]. Applications to experiments on repetitive movements [Dotov & Frank, 2011; Dotov et al., 2015; Frank et al., 2013; Gordon et al., 2016; Kim et al., 2015] and oscillatory systems described by Nambu dynamics [Chaikhan et al., 2016; Frank, 2010a,b, 2017; Mathis & Mathis, 2014; Mathis et al., 2013; Mongkolsakulvong et al., 2012] can be found that address not only exact, analytical approaches but also approximate solution methods [Weber et al., 2018].

While the concept of active systems has been used to describe synchronization phenomena in swarming and foraging (see references above), at issue is how to utilize the canonical-dissipative approach to obtain insights into the synchronization of coupled self-oscillators, see Fig. 1 (right hand side). In this context, the current study focuses on multifrequency or polyrhythmic synchronization in which the oscillators synchronize their dynamics while oscillating with different frequencies. Polyrhythmic synchronization can be observed in musicians (e.g. drummers and piano players [Peper et al., 1995]). Recent studies suggest that synchronized polyrhythmic brain activity plays a crucial role for the functioning of the human and animal brain [Battaglia & McNaughton, 2011; Fujisawa & Buzsaki, 2011; Stupacher et al., 2017]. In fact, in a series of previous studies, coupled Rayleigh oscillators have been discussed within the framework of active systems featuring negative friction and in which both the monofrequency and multifrequency cases have been considered [Ebeling et al., 1999; Erdmann & Ebeling, 2005; Erdmann et al., 2000; Schimansky-Geier et al., 2005] (i.e., the perspective in the upper right corner in Fig. 1 has been used). However, in these studies a strict canonical-dissipative perspective based on coupled CDOs has not been used. In contrast, the goal of the present study is to develop a framework for monofrequency and multifrequency synchronization...
Multifrequency synchronization in canonical dissipative systems

Fig. 1. Previous work related to the current study. The topic of "coupled canonical-dissipative oscillators" in the lower right corner highlighted with dashed lines has received relatively little attention despite the fact that the canonical-dissipative approach, in general, has the benefit to allow for semi-analytical approaches that are often difficult to achieve otherwise.

of coupled oscillators that is strictly based on coupled canonical-dissipative oscillators (i.e., to address synchronization from the perspective listed in lower right corner in Fig. 1). In particular, bifurcation diagrams characterizing the multifrequency limit cycle attractors and semi-analytical expressions describing the attractors themselves will be derived, see Secs. 2 and 3. The current work is based on previous works by Holt [1982] on 1:2 multifrequency oscillations (Sec. 3.2) and Fokas & Lagerstrom [1980] on 1:3 multifrequency oscillations (Sec. 3.3) in Hamiltonian systems.

2. General considerations

2.1. The fundamental CDO

We briefly review the fundamental CDO model [Ebeling & Sokolov, 2004; Feistel & Ebeling, 1989; Frank, 2005, 2010c]. Let \( q \) and \( p \) denote the position and momentum variables of an oscillator. \( H \) denotes the Hamiltonian function defined by

\[
H = \frac{p^2}{2m} + \frac{Kq^2}{2}.
\]

In the context of an ordinary harmonic oscillator the two parameters \( m > 0 \) and \( K > 0 \) correspond to the mass of a particle and a spring constant. In general, \( m \) and \( K \) denote two model parameters that (as we will show) determine the dynamics of the fundamental CDO on its limit cycle. The Hamiltonian evolution equations of an ordinary harmonic oscillator read

\[
\frac{d}{dt}p = -\frac{\partial H}{\partial q}, \quad \frac{d}{dt}q = \frac{\partial H}{\partial p},
\]

where \( t \) denotes time and \( H \) is given by Eq. (1). They describe the evolution of an harmonic oscillator with angular frequency \( \omega = \sqrt{K/m} \). The Hamiltonian function \( H \) is an invariant of the oscillator dynamics. In contrast, the fundamental CDO is a nonlinear oscillator that (i) is a limit cycle oscillator (i.e., exhibits a limit cycle attractor) and (ii) exhibits a limit cycle that is described by the Hamiltonian evolution equations (2). Explicitly, the canonical-dissipative oscillator is defined by [Ebeling & Sokolov, 2004; Frank, 2010c]

\[
\frac{d}{dt}p = -\frac{\partial H}{\partial q} - \frac{\partial G}{\partial p}, \quad \frac{d}{dt}q = \frac{\partial H}{\partial p}
\]
with a G-function [Ebeling, 2004; Schweitzer et al., 2001] that reads

\[ G = \gamma \frac{(H - B)^2}{2}. \]  

(4)

The function G describes both a pumping and damping mechanism. Comparing Eqs. (2) and (3), we see that Eq. (3) is a generalization of Eq. (2) and includes Eq. (2) for the special case \( \gamma = 0 \). The parameter \( \gamma \geq 0 \) is a coupling parameter of the pumping and damping mechanism. From Eq. (3) it follows that \( \frac{dG}{dt} \leq 0 \) (see e.g. Ebeling & Sokolov [2004]; Frank [2010c]; Gordon et al. [2016]). Consequently, for \( B \leq 0 \) we have \( G \rightarrow B^2/2 \) and \( H \rightarrow 0 \) in the limit case \( t \rightarrow \infty \), which implies that the point \( (q, p) = (0, 0) \) corresponds to a stable fixed point. In contrast, for \( B > 0 \) we have \( G \rightarrow 0 \), which implies \( H \rightarrow B \) for \( t \rightarrow \infty \). Therefore, \( (q, p) = (0, 0) \) is an unstable fixed point. In particular, any trajectory with initial condition \( (q, p) \neq (0, 0) \) converges to the limit cycle defined by \( q(t) = r \sin(\omega t + h) \) with \( r = \sqrt{2B/K} \) and \( h \) being arbitrary. Consequently, the parameter \( B \) can be considered as a bifurcation parameter. At \( B = 0 \) a Hopf bifurcation occurs. From a mechanistic point of view, the parameter \( B \) describes an effective pumping parameter. That is, \( B \) describes the difference between the coefficients of a linear pumping force and a linear damping force [Frank, 2010d]. If the pumping outweighs the damping, we have \( B > 0 \); otherwise \( B \leq 0 \). Note that the function \( H \) is generally not an invariant of the dynamics of the CDO. However, it is an invariant of the dynamics for the special case when the CDO evolves on its limit cycle. Therefore, \( H \) will be referred to as pseudo-invariant. The aforementioned Hopf bifurcation can be conveniently illustrated by means of the amplitude equation of the fundamental CDO. Let us define the complex-valued amplitude \( A \) of the CDO by \( q(t) = A(t) \exp\{i\omega t\} + c.c. \). Assuming that \( \gamma \) is a small parameter (more precisely the characteristic time constant \( \tau = 1/\gamma \) is assumed to be large relative to the oscillation period \( T = 2\pi/\omega \) such that \( \tau \gg T \)), from Eq. (3) we obtain the amplitude equation \( \frac{dA}{dt} = \gamma \omega^2 A \left( \frac{B}{2K} - |A|^2 \right) \). [Mongkolsakulvong & Frank, 2017] using the rotating wave approximation and the slowly varying amplitude approximation [Haken, 1985]. Taking only first order terms of \( \gamma \) into account, we have

\[ \frac{dA}{dt} = \gamma \omega^2 A \left( \frac{B}{2K} - |A|^2 \right). \]

Consequently, for \( B \leq 0 \) the amplitude dynamics approaches \( |A| = 0 \), while for \( B > 0 \) we have \( |A| \rightarrow \sqrt{B/(2K)} \) for \( t \rightarrow \infty \), which is consistent with the considerations made above regarding the Hopf bifurcation.

### 2.2. Coupled CDOs

Let us consider two CDOs \( j = 1, 2 \) with position and momentum variables \( q_j \) and \( p_j \) and Hamiltonian functions \( H_j \) such that \( H_1 = H_1(q_1, p_1) \) and \( H_2 = H_2(q_2, p_2) \) holds. That is, the functions only depend on the variables of the respective oscillators. In the uncoupled conservative case the oscillators satisfy the Hamiltonian equations

\[ \frac{dp_j}{dt} = -\frac{\partial H_j}{\partial q_j}, \quad \frac{dq_j}{dt} = \frac{\partial H_j}{\partial p_j}, \]

(6)

such that \( H_1 \) and \( H_2 \) are invariants of the dynamics. In what follows, we consider the case in which for the dynamics of the conservative case defined by Eq. (6) there exist three invariants \( I_k \) with \( k = 1, 2, 3 \). The first two invariants are given by \( I_k = H_k \) with \( k = 1, 2 \). This implies that \( I_1 \) and \( I_2 \) depend only on the variables of the respective oscillators \( j = 1 \) and \( j = 2 \). In contrast, the third invariant \( I_3 \) is assumed to depend on momentum and position variables of both oscillators (e.g., \( I_3 = I_3(q_1, q_2), I_3 = I_3(p_1, p_2), \ldots, I_3 = I_3(q_1, q_2, p_1, p_2) \)). Therefore, it can be used to couple the CDOs. Let us introduce the total Hamiltonian function \( H_{\text{tot}} = H_1 + H_2 \). From \( I_k = \text{const} \) for solutions \( q_1(t), q_2(t), p_1(t), p_2(t) \) of Eq. (6) it follows that

\[ \frac{dI_k}{dt} = 0 \Rightarrow \sum_{j=1}^{2} \left( \frac{\partial I_k}{\partial q_j} \frac{\partial H_{\text{tot}}}{\partial p_j} - \frac{\partial I_k}{\partial p_j} \frac{\partial H_{\text{tot}}}{\partial q_j} \right) = 0 \]

(7)
holds. Let us define the term in Eq. (7) as function $S$ like

$$S_k(q_1, q_2, p_1, p_2) = \sum_{j=1}^{2} \left( \frac{\partial I_k}{\partial q_j} \frac{\partial H_{tot}}{\partial p_j} - \frac{\partial I_k}{\partial p_j} \frac{\partial H_{tot}}{\partial q_j} \right).$$

Then, we have $S = 0$ for $q_1(t), q_2(t), p_1(t), p_2(t)$ defined by Eq. (6). The function $S$ will be used in the proof given in Eq. (13) below. We are now in the position to introduce two uncoupled CDOs as generalizations of Eq. (6) like

$$\frac{d}{dt} p_j = -\frac{\partial H_j}{\partial q_j} - \frac{\partial G_j}{\partial p_j}, \quad \frac{d}{dt} q_j = \frac{\partial H_j}{\partial p_j}$$

with

$$G_1 = \frac{I_1 - B_1}{2} = \frac{(H_1 - B_1)^2}{2}, \quad G_2 = \frac{I_2 - B_2}{2} = \frac{(H_2 - B_2)^2}{2}.$$  \hspace{1cm} (10)

By analogy to our considerations in Sec. 2.1, the functions $G_k$ with $k = 1, 2$ become stationary in the long time limit such that $\partial G_j/\partial p_j = 0$. That is, Eq. (9) reduces to Eq. (6) for $t \to \infty$. Therefore, for the uncoupled CDOs all three functions $I_k$ become invariants in the limiting case $t \to \infty$. Let us assume that for $H_1$ and $H_2$ there exist lower boundaries $H_{1, \text{min}}$ and $H_{2, \text{min}}$, respectively, with $\forall q_1, q_2, p_1, p_2 : H_1 \geq H_{1, \text{min}}, H_2 \geq H_{2, \text{min}}$, where $H_{1, \text{min}}$ and $H_{2, \text{min}}$ denote the largest values of such boundaries. Then, for $B_1 > H_{1, \text{min}}$ and $B_2 > H_{2, \text{min}}$ the Hamiltonian functions $H_1$ and $H_2$ converge to the parameters $B_1$ and $B_2$, respectively, such that the CDOs (9) become ordinary, conservative oscillators of the form (6), whose Hamiltonian functions assume the specific values $B_1$ and $B_2$. Having introduced two uncoupled CDOs, let us couple the CDOs by means of the third invariant $I_3$. Accordingly, the coupled CDO system reads

$$\frac{d}{dt} p_j = -\frac{\partial H_{tot}}{\partial q_j} - \frac{\partial G_{tot}}{\partial p_j}, \quad \frac{d}{dt} q_j = \frac{\partial H_{tot}}{\partial p_j}$$

with

$$G_k = \frac{(I_k - B_k)^2}{2} \text{ for } k = 1, 2, 3, \quad G_{\text{tot}} = \sum_{k=1}^{3} G_k,$$  \hspace{1cm} (12)

where we have introduced the total $G$-function $G_{\text{tot}}$. The coupled CDO model defined by Eq. (11) includes the model (9) of the two uncoupled CDOs as a special case for $\gamma_3 = 0$. In order to see that Eq. (11) reduces to Eq. (9) for $\gamma_3 = 0$ let us just note that $\partial H_{\text{tot}}/\partial q_j = \partial H_j/\partial q_j$, $\partial H_{\text{tot}}/\partial p_j = \partial H_j/\partial p_j$, and $\partial G_{\text{tot}}/\partial p_j = \partial G_j/\partial p_j + \partial G_3/\partial p_j$ holds for $j = 1, 2$. Consequently, the coupled CDO model (11) differs from the model (9) of two uncoupled CDOs only by the additional terms $-\partial G_3/\partial p_j$ in the evolution equations for momentum variables $p_j$. In this context, note that $H_{\text{tot}} = H_1 + H_2 = \sum_{k=1,2} I_k$, whereas $G_{\text{tot}} = \sum_{k=1}^{3} G_k$. This implies that Eq. (11b) reads $dq_j/dt = \partial H_j/\partial p_j$. That is, in the case of the coupled oscillators (11), the relationships between the velocity variables $dq_j/dt$ and the momentums $p_j$ are the same as for the uncoupled oscillators (9).

Let us address some general properties of the coupled CDO model (11). First, let us determine the
evolution of $G_{\text{tot}}$. To this end, we calculate

$$\frac{d}{dt} G_{\text{tot}} = \sum_{k=1}^{3} \frac{\partial G_{\text{tot}}}{\partial I_k} \frac{d}{dt} I_k$$

$$= \sum_{k=1}^{3} \frac{\partial G_{\text{tot}}}{\partial I_k} \left[ \sum_{j=1,2} \left( \frac{\partial I_k}{\partial q_j} \frac{\partial H_{\text{tot}}}{\partial p_j} - \frac{\partial I_k}{\partial p_j} \frac{\partial H_{\text{tot}}}{\partial q_j} \right) \right]$$

$$= \sum_{k=1}^{3} \frac{\partial G_{\text{tot}}}{\partial I_k} \left[ S_k(q_1, q_2, p_1, p_2) - \sum_{j=1,2} \frac{\partial I_k}{\partial p_j} \frac{\partial G_{\text{tot}}}{\partial p_j} \right] = - \sum_{j=1,2} \frac{\partial G_{\text{tot}}}{\partial p_j} \left( \sum_{k=1}^{3} \frac{\partial G_{\text{tot}}}{\partial I_k} \frac{\partial I_k}{\partial p_j} \right)$$

$$= - \sum_{j=1,2} \left( \frac{\partial G_{\text{tot}}}{\partial p_j} \right)^2 \leq 0 . \quad (13)$$

Consequently, $G_{\text{tot}}$ is a monotonically decreasing function over time. Let $G_{\text{tot,min}}$ denote the minimal value that $G_{\text{tot}}$ can assume in the four-dimensional space spanned by the position and momentum variables. By definition of $G_{\text{tot}}$, see Eq. (12), we have $G_{\text{tot}} \geq G_{\text{tot,min}} \geq 0$. Given that there exists a lower bound for $G_{\text{tot}}$, from Eq. (13) it follows that $dG_{\text{tot}}/dt = 0$ for $t \to \infty$, that is, $G_{\text{tot}}$ becomes stationary. If $G_{\text{tot}}$ becomes stationary the individual terms $G_k$ may become stationary as well. If so, then $I_k$ become invariants of the dynamics of the coupled CDOs in the long time limit. In particular, if the functions $I_k$ converge individually to their respective $B$-parameters such that $I_1 = H_1 = B_1$, $I_2 = H_2 = B_2$, and $I_3 = B_3$, then Eq. (11) reduces to Eq. (6) and the coupled CDOs exhibit the same dynamics as the ordinary, conservative uncoupled Hamiltonian oscillators provided that we choose the initial conditions appropriately. In other words, for $I_k = B_k$ with $k = 1, 2, 3$ the function $I_3$ is an invariant of the dynamics of the coupled CDOs (11) just like the Hamiltonian functions $H_1 = I_1$ and $H_2 = I_2$ are invariants in that special case. That being said, the functions $I_k$ are generally not invariants of the dynamics defined by Eq. (11). Therefore, the functions $I_k$ will be referred to as pseudo-invariants.

The aim of the current study is to examine multifrequency synchronization in coupled CDO models of the form (11) by means of amplitude equations. Accordingly, let us assume that $G_{\text{tot}}$ becomes stationary and that for this case the dynamics of the coupled CDOs (11) approaches a limit cycle attractor in the full four-dimensional phase space spanned by $q_1, q_2, p_1, p_2$. In order to derive amplitude equations, we assume that the parameters $\gamma_k$ are relatively small like $\gamma_k = \tilde{G}_k \epsilon$, where $\epsilon$ is a common small parameter. Let us define the oscillator amplitudes $A_j$ for the two CDOs $j = 1, 2$ like

$$q_j(t) = A_j(t) \exp \{ i \omega_j t \} + c.c. \quad (14)$$

Then, in the sections below, amplitude equations will be derived of the form

$$\frac{d}{dt} A_j = N_j^{(1)}(A_1, A_2) + O(\epsilon^2) , \quad (15)$$

where $N_j^{(1)}$ are certain functions that depend on the details of the functions $H_1$ ($I_1$), $H_2$ ($I_1$), and $I_3$. For the purposes of the current study it will be sufficient to consider only the first order terms in $\epsilon$ as indicated by the super-index (1) in $N_j^{(1)}$. In order to derive the first-order terms $N_j^{(1)}$, it is sufficient to use the aforementioned rotating wave and slowly varying amplitude approximation methods [Haken, 1985], which yield in the order $\epsilon$ consistent results with other methods such as averaging methods [O’Malley Jr. & Williams, 2006; Verhulst, 1996]. In general, Eq. (15) will be derived in two steps. In a first step, we will express the three pseudo-invariants $I_k$ in terms of the two amplitudes $A_j$ such that we obtain

$$I_k = F_k(A_k) \text{ for } k = 1, 2 \text{ , } I_3 = F_3(A_1, A_2) . \quad (16)$$

To this end, the momentum variables $p_j$ for $j = 1, 2$ in $I_k$ will be replaced in terms of the variables $q_j$ and $dq_j/dt$ using $dq_j/dt = \partial H_j/\partial p_j$. Having $I_k$ expressed in the from of Eq. (16), in a second step, we proceed to write the first order Hamiltonian equations for $q_j$ and $p_j$ in terms of second order Newtonian oscillator
In this case, Eq. (11) reads
\[
\frac{d^2}{dt^2} q_j = U_j^{(0)} (p_j, q_j) + \epsilon U_j^{(1)} (q_1, q_2, p_1, p_2, I_j(A_j), I_3(A_1, A_2)) + O(\epsilon^2) .
\] (17)
Again, the momentum variables \( p_j \) will be replaced in terms of position variables \( q_j \) and "velocity" variables \( dq_j/dt \) using \( dq_j/dt = \partial H_j/\partial p_j \). In these second order equations the pseudo-invariants will be replaced by the corresponding amplitude functions as indicated in Eq. (17). Using Eqs. (14) and (17) the amplitude equations of the form (15) will be derived.

### 2.3. Monofrequency synchronization of two coupled standard CDOs

#### 2.3.1. Model

As will be shown below, the multifrequency case can be treated similar to the monofrequency case. Therefore, we first discuss the monofrequency case, the case in which two coupled oscillators exhibit the same oscillation frequency. We begin by considering two fundamental CDOs with
\[
H_j = \frac{p_j^2}{2m} + Kq_j^2 .
\] (18)
The uncoupled CDO model satisfies Eq. (9). In the uncoupled case, for \( B_1, B_2 > 0 \) the dynamics of the uncoupled oscillators converges to limit cycle attractors described by \( q_j(t) = A_{j,st} \exp \{ i \omega_j t \} + c.c. \), the frequencies \( \omega_1 = \omega_2 = \omega_0 = \sqrt{K/m} \), and certain stationary complex-valued amplitudes \( A_{j,st} \) with \( |A_{j,st}| > 0 \). It can be shown (see, e.g., Mongkolsakulvong & Frank [2017]) that the third invariant \( I_3 \) is the ordinary angular momentum \( L_0 \) defined by
\[
L_0 = q_1 p_2 - q_2 p_1 .
\] (19)
Accordingly, let us define the coupled CDO model for monofrequency synchronization by Eq. (11) with
\[
I_1 = H_1 , \; I_2 = H_2 , \; I_3 = L_0 .
\] (20)
In this case, Eq. (11) reads
\[
\frac{d}{dt} p_1 = -Kq_1 - \gamma_1 \frac{p_1}{m} (H_1 - B_1) + \gamma_3 q_2 (L_0 - B_3) , \quad \frac{d}{dt} q_1 = \frac{p_1}{m} ,
\]
\[
\frac{d}{dt} p_2 = -Kq_2 - \gamma_2 \frac{p_2}{m} (H_2 - B_2) - \gamma_3 q_1 (L_0 - B_3) , \quad \frac{d}{dt} q_2 = \frac{p_2}{m} .
\] (21)

#### 2.3.2. Pseudo-invariants and monofrequency-synchronization limit cycle attractors

First, let us determine the pseudo-invariants \( I_k \) as functions of \( A_j \). From Eq. (14), \( p_j = m dq_j/dt \), and by means of the slowly varying amplitude approximation [Haken, 1985], we obtain
\[
p_j = m i \omega_j A_j \exp \{ i \omega_j t \} + c.c. \tag{22}
\]
Substituting Eqs. (14) and (22) into the invariants listed in Eq. (20) we obtain
\[
I_1 = 2K |A_1|^2 , \; I_2 = 2K |A_2|^2 , \; I_3 = 4 \omega_0 m \Re(e^{iA_1^* A_2}) ,
\] (23)
where \( ^* \) denotes the complex conjugate and \( \Re(\cdot) \) denotes the real part. Next, we express the complex-valued amplitudes \( A_j \) in polar coordinates using the the phases \( \phi_j \) and amounts \( R_j = |A_j| \) like \( A_j = R_j e^{i \phi_j} \). Then Eq. (23) reads
\[
I_1 = 2K R_1^2 , \; I_2 = 2K R_2^2 , \; I_3 = 4 \omega_0 m R_1 R_2 \sin(\phi_1 - \phi_2) .
\] (24)
From Eq. (21) and our considerations in Sec. 2.2 it follows that under appropriate conditions there exist limit cycle attractors with \( G_{tot} = 0 \) and \( I_k = B_k \). In other words, we have
\[
I_1 = H_1 = B_1 \Rightarrow R_1 = \sqrt{B_1/(2K)} , \; I_2 = H_2 = B_2 \Rightarrow R_2 = \sqrt{B_2/(2K)} ,
\]
\[
I_3 = L_0 = B_3 \Rightarrow R_3 = 4 \omega_0 m R_1 R_2 \sin(\phi_1 - \phi_2) \tag{25}
\]
Limit cycle attractors of this kind only exists when all three conditions listed in Eq. (25) are satisfied simultaneously with \( H_1, H_2 > 0 \). This implies \( B_1, B_2 > 0 \). Moreover, for \( I_1 = B_1 > 0 \) and \( I_2 = B_2 > 0 \) the RHS factor \( \xi \) introduced in Eq. (25) reads \( \xi = 4\omega_0 m R_1(B_1)R_2(B_2) = 2\sqrt{B_1B_2}/\omega_0 \). Since the sine function in Eq. (25c) is bounded in the interval \([-1,1]\) the total right-hand side of Eq. (25c) cannot be larger in the amount than \( \xi = 2\sqrt{B_1B_2}/\omega_0 \). Therefore, limit cycle attractors with \( I_k = B_k \) for \( k = 1, 2, 3 \) exist only if \( B_3 \) is not too large in the amount. More precisely, a critical value \( B_{3,\text{crit}} \) for \( B_3 \) can be defined by

\[
B_{3,\text{crit}} = \frac{2}{\omega_0}\sqrt{B_1B_2}.
\]

Then, limit cycle attractors with \( I_k = B_k \) and \( k = 1, 2, 3 \) exist if \( B_3 \in [-B_{3,\text{crit}}, B_{3,\text{crit}}] \). Let us introduce the relative difference (or phase difference)

\[
\psi = \phi_1 - \phi_2.
\]

If \( B_3 \in [-B_{3,\text{crit}}, B_{3,\text{crit}}] \) holds and the system is on the limit cycle with \( I_k = B_k \) for \( k = 1, 2, 3 \), then we obtain a stationary relative phase \( \psi_{st} \) given by

\[
B_3 = \frac{2}{\omega_0}\sqrt{B_1B_2}\sin(\psi_{st}) = B_{3,\text{crit}}\sin(\psi_{st}).
\]

Let us introduce the auxiliary variable (that will be identified as a bifurcation parameter below)

\[
\eta = \frac{B_3}{B_{3,\text{crit}}} = \frac{\omega_0 B_3}{2\sqrt{B_1B_2}}.
\]

In terms of the variable \( \eta \), the condition \( B_3 \in [-B_{3,\text{crit}}, B_{3,\text{crit}}] \) is equivalent to the condition \( \eta \in [-1,1] \). If \( \eta \in [-1,1] \) holds, we have

\[
\eta = \sin(\psi_{st}) \Rightarrow \psi_{st} = \arcsin(\eta).
\]

The fact that \( I_k = B_k \) with \( k = 1, 2, 3 \) gives us \( G_{tot} = G_{tot, min} = 0 \) and in general we have \( dG_{tot}/dt \leq 0 \) (see Sec. 2.2) indicates that the orbit described by \( I_k = B_k \) and \( k = 1, 2, 3 \) is indeed a stable limit cycle and not an unstable orbit.

2.3.3. Amplitude equations

In order to discuss the stability of the limit cycle defined by \( I_k = B_k \) with \( k = 1, 2, 3 \) in more detail as well as the cases \( B_3 < -B_{3,\text{crit}} \) and \( B_3 > B_{3,\text{crit}} \), we determine the amplitude equations of the coupled CDO model (21) for monofrequency synchronization. From Eq (21) we obtain second order Newtonian oscillator equations of the form (17) that read

\[
\frac{d^2}{dt^2} q_1 = -\frac{K}{m} q_1 - \gamma_1 \frac{p_1}{m^2} (H_1(A_1) - B_1) + \frac{\gamma_3}{m} q_2 (L_0(A_1, A_2) - B_3),
\]

\[
\frac{d^2}{dt^2} q_2 = -\frac{K}{m} q_2 - \gamma_2 \frac{p_2}{m^2} (H_2(A_2) - B_2) - \frac{\gamma_3}{m} q_1 (L_0(A_1, A_2) - B_3).
\]

Substituting Eqs. (14) and (22) into Eq. (31) and using the rotating wave and slowly varying amplitude approximation methods [Haken, 1985], we obtain

\[
\frac{d}{dt} A_1 = \frac{\gamma_1}{2m} A_1 (H_1(A_1) - B_1) - i \frac{\gamma_3}{2\omega_0 m} A_2 (L_0(A_1, A_2) - B_3),
\]

\[
\frac{d}{dt} A_2 = \frac{\gamma_2}{2m} A_2 (H_2(A_2) - B_2) + i \frac{\gamma_3}{2\omega_0 m} A_1 (L_0(A_1, A_2) - B_3).
\]

Expressing the amplitudes in polar coordinates (see above), we obtain

\[
\frac{d}{dt} R_1 = \frac{\gamma_1}{2m} R_1 (H_1(R_1) - B_1) - \frac{\gamma_3}{2\omega_0 m} R_2 \sin(\psi) (L_0(R_1, R_2, \psi) - B_3),
\]

\[
\frac{d}{dt} R_2 = \frac{\gamma_2}{2m} R_2 (H_2(R_2) - B_2) - \frac{\gamma_3}{2\omega_0 m} R_1 \sin(\psi) (L_0(R_1, R_2, \psi) - B_3).
\]
and for \( R_1, R_2 > 0 \)

\[
\begin{align*}
\frac{d}{dt} \phi_1 &= -\frac{\gamma_3}{2\omega_0m} \frac{R_2}{R_1} \cos(\psi)(L_0(R_1, R_2, \psi) - B_3), \\
\frac{d}{dt} \phi_2 &= \frac{\gamma_3}{2\omega_0m} \frac{R_1}{R_2} \cos(\psi)(L_0(R_1, R_2, \psi) - B_3).
\end{align*}
\]  

(34)

Since the evolution equations for \( \phi_j \) do only depend on the relative phase \( \psi \), we obtain an evolution equation for \( \psi \) that only depends on \( \psi, R_1, \) and \( R_2 \):

\[
\frac{d}{dt} \psi = -\frac{\gamma_3}{2\omega_0m} \left( \frac{R_1}{R_2} + \frac{R_2}{R_1} \right) \cos(\psi)(L_0(R_1, R_2, \psi) - B_3).
\]

(35)

Equations (33) and (35) are a closed set of differential equations for \( R_1, R_2, \) and \( \psi \).

### 2.3.4. Semi-analytical stability analysis and bifurcation diagram

In this section, a semi-analytical stability analysis is conducted based only on the dynamics of the relative phase \( \psi \), ignoring the dynamics of the amplitudes \( R_1 \) and \( R_2 \). The semi-analytical analysis will be presented in some detail. We do so because the arguments presented here for the case of monofrequency synchronization between CDOs will also be used to discuss the case of multifrequency synchronization between CDOs in Sec. 3. That is, in Sec. 3 we will refer to the arguments that will be developed in this section. Moreover, the results of the semi-analytical analysis will be supported below by numerical simulations.

Let us consider the special case \( B_1, B_2 > 0 \) and \( B_3 = 0 \) \( \Rightarrow \eta = 0 \). Using \( L_0 = 4m\omega_0 R_1 R_2 \sin(\psi) \) (see Eqs. (24c)), from Eq. (35) we obtain

\[
\frac{d}{dt} \psi = -\gamma_3 \left( \frac{R_1^2}{R_2} + \frac{R_2^2}{R_1} \right) \sin(2\psi)
\]

(36)

with \( R_1, R_2 > 0 \). Consequently, the dynamics of \( \psi \) exhibits fixed points at \( \psi_{st} = 0, \pi/2, \pi, 3\pi/2 \). Assuming that \( R_{1,2} \) are fixed (see our introductory comment), fixed points at \( \psi_{st} = 0, \pi \) are asymptotically stable, while fixed points at \( \psi_{st} = \pi/2, 3\pi/2 \) are unstable. With respect to limit cycle attractors, we find for \( B_1, B_2 > 0 \) and \( B_3 = 0 \) there are two stable limit cycles with either \( \psi_{st} = 0 \) or \( \psi_{st} = \pi \) and \( I_k = B_k \) for \( k = 1, 2 \) such that

\[
q_1(t) = r_1 \cos(h + \psi_{st}), \quad q_2(t) = r_2 \cos(h)
\]

(37)

with arbitrary phase \( h \), \( \psi_{st} = 0, \pi \), and \( r_j = 2B_j = \sqrt{2B_j/K} \) for \( j = 1, 2 \). The two limit cycle attractors may be interpreted as in-phase (\( \psi_{st} = 0 \)) and anti-phase (\( \psi_{st} = \pi \)) attractors, respectively. In addition, there are two unstable closed orbits given by Eq. (37) with arbitrary phase \( h \), \( \psi_{st} = \pi/2, 3\pi/2 \), and \( r_{1,2} = \sqrt{2B_{1,2}/K} \).

Comparing the results obtained here by use of amplitude equations with the results given in the previous section by Eq. (30) from the analysis of pseudo-invariants, we see that the analysis of the pseudo-invariants yields for \( B_3 = 0 \) only the fixed points \( \psi_{st} = 0, \pi \) of the stable limit cycles and does not account for the unstable closed orbits with \( \psi_{st} = \pi/2, 3\pi/2 \). The reason for this is that the fixed points \( \psi_{st} = \pi/2, 3\pi/2 \) only occur for \( \eta = \pm 1 \) in the context of limit cycles with \( I_k = B_k \) for \( k = 1, 2, 3 \). However, the condition \( \eta = \pm 1 \Rightarrow B_3 = \pm B_{3,crit} \) is inconsistent with \( B_3 = 0 \). In short, the unstable closed orbits predicted by the amplitude equations are orbits for which \( I_1 = B_1 \) and \( I_2 = B_2 \) hold but \( I_3 = L_0(\psi = \pi/2, 3\pi/2) = \pm 2\sqrt{B_1B_2}/\omega_0 \neq B_3 \).

Let us consider the case with \( B_3 \neq 0 \) and \( B_3 \in [-B_{3,crit}, B_{3,crit}] \) (i.e., \( \eta \neq 0 \) and \( \eta \in [-1, 1] \)). We consider the fixed points with \( I_k = B_k \) for \( k = 1, 2, 3 \) which implies \( \eta = \sin(\psi_{st}) \). For \( \eta \in (0, 1) \) we have two fixed points with \( \eta = \sin(\psi_{st}) \) and \( \psi_{st} \in (0, \pi) \) that are symmetric with respect to \( \pi/2 \), while for \( \eta \in (-1,0) \) we have two fixed points with \( \eta = \sin(\psi_{st}) \) and \( \psi_{st} \in (\pi, 2\pi) \) that are symmetric with respect to \( 3\pi/2 \). Let us consider \( \eta \in (0, 1) \) and \( \psi_{st} \) to the left of \( \pi/2 \) (i.e., \( \psi_{st} \in (0, \pi/2) \)). For a small perturbation \( \delta > 0 \) we have \( \psi = \psi_{st} + \delta \) and \( \sin(\psi) > \sin(\psi_{st}) \), which implies \( L_0(\psi) > L_0(\psi_{st}) \). However \( L_0(\psi_{st}) = B_3 \) holds. Therefore, we get \( L_0(\psi) > B_3 \Rightarrow L_0(\psi) - B_3 > 0 \). Moreover, we have \( \cos(\psi_{st}) > 0 \).
and, for sufficiently small perturbations, $\cos(\psi) > 0$. In this case, from Eq. (35) it follows that $d\psi/dt < 0$. In summary, we conclude

$$\eta \in (0,1), \eta = \sin(\psi_{st}), \psi_{st} \in (0, \pi/2), \psi = \psi_{st} + \delta, \delta > 0$$

$$\Rightarrow L_0(\psi) - B_3 > 0, \cos(\psi) > 0 \overset{\text{Eq(35)}}{\Rightarrow} \frac{d}{dt}\psi < 0 . \quad (38)$$

By analogy, we arrive at the conclusion

$$\eta \in (0,1), \eta = \sin(\psi_{st}), \psi_{st} \in (0, \pi/2), \psi = \psi_{st} + \delta, \delta < 0$$

$$\Rightarrow L_0(\psi) - B_3 < 0, \cos(\psi) > 0 \overset{\text{Eq(35)}}{\Rightarrow} \frac{d}{dt}\psi > 0 . \quad (39)$$

Analogous considerations can be made for $\eta \in (0,1)$ and $\psi_{st} \in (\pi/2, \pi)$ and for $\eta \in (-1,0)$ with $\psi_{st} \in (\pi, 3\pi/2)$ or $\psi_{st} \in (3\pi/2, 2\pi)$. In all cases we find that if $\delta > 0$ then $d\psi/dt < 0$ holds and vice versa if $\delta < 0$ then $d\psi/dt > 0$ holds. Therefore, within the context of the relative phase dynamics (i.e., within the framework of our semi-analytical approach) the fixed points $\psi_{st}$ defined by $I_3 = B_3 \Rightarrow \eta = \sin(\psi_{st})$ are stable fixed points. This suggests that the corresponding closed orbits in the four dimensional space $q_1, q_2, p_1, p_2$ are limit cycle attractors.

Fixing the $R_1$ and $R_2$ variables, Eq. (35) exhibits fixed points at $\psi_{st} = \pi/2, 3\pi/2$ for which $\cos(\psi_{st}) = 0$ holds. In line with our considerations above, there are closed orbits with relative phases $\psi_{st} = \pi/2, 3\pi/2, I_1 = B_1, I_2 = B_2$ but $I_3 \neq B_3$. In particular, we have

$$L_0(\psi = \pi/2, 3\pi/2) = \pm \frac{2}{\omega} \sqrt{B_1 B_2} = \pm B_{3,\text{crit}} . \quad (40)$$

Let us consider $\eta \in (0,1) \Rightarrow B_3 \in (0, B_{3,\text{crit}})$. Let us consider $\psi_{st} = \pi/2$ and a perturbation $\psi = \pi/2 + \delta$, where $\delta > 0$ is small. Then $\cos(\psi) < 0$ holds. In addition, for sufficiently small values of $\delta$ we have $L_0(\psi_{st} = \pi/2) = B_{3,\text{crit}} \Rightarrow B_3 \Rightarrow L_0(\psi_{st} + \delta) > B_3 \Rightarrow L_0(\psi) - B_3 > 0$. From Eq. (35) it then follows that $d\psi/dt > 0$. In summary, we conclude that

$$\eta \in (0,1), \psi_{st} = \pi/2, \psi = \psi_{st} + \delta, \delta > 0$$

$$\Rightarrow L_0(\psi) - B_3 > 0, \cos(\psi) < 0 \overset{\text{Eq(35)}}{\Rightarrow} \frac{d}{dt}\psi > 0 . \quad (41)$$

By analogy, we arrive at the conclusion

$$\eta \in (0,1), \psi_{st} = \pi/2, \psi = \psi_{st} + \delta, \delta < 0$$

$$\Rightarrow L_0(\psi) - B_3 < 0, \cos(\psi) > 0 \overset{\text{Eq(35)}}{\Rightarrow} \frac{d}{dt}\psi < 0 . \quad (42)$$

Note that those relations also hold for $\eta \in (-1,0)$. That is, Eqs. (41) and (42) also hold if we replace $\eta \in (0,1)$ by $\eta \in (-1,0)$. In this case $B_3 < 0$ holds rather than $B_3 > 0$. However, the inequality $L_0(\psi) - B_3 > 0$ holds for $\psi = \pi/2 + \delta$ with $\delta$ sufficiently small (and positive or negative) in any case irrespective of the sign of $B_3$ as long as $\eta \in (-1,1)$. Moreover, analogous consideration can be made for $\psi_{st} = 3\pi/2$. In all those cases we find that that $\delta > 0$ implies $d\psi/dt > 0$ and vice versa $\delta < 0$ implies $d\psi/dt < 0$. Therefore, within the context of the relative phase dynamics the fixed points $\psi_{st} = \pi/2, 3\pi/2$ are unstable fixed points. This suggests that the corresponding orbits given by Eq. (37) with arbitrary phase $h, \psi_{st} = \pi/2, 3\pi/2$, and $r_{1,2} = \sqrt{2B_{1,2}/K}$ not only for $B_3 = 0$ (i.e., $\eta = 0$) but also for $B_3 \in (-B_{3,\text{crit}}, B_{3,\text{crit}})$ (i.e., $\eta \in (-1,1)$) in the four dimensional space $q_1, q_2, p_1, p_2$ are unstable close orbits.

From these considerations, we can draw the bifurcation diagram shown in Fig. 2. The bifurcation diagram shows the possible stationary values $\psi_{st}$ for a given bifurcation parameter $\eta$ that characterize the limit cycle attractors (multistable case) or limit cycle attractor (monostable case) that exist for the selected value $\eta$. The solid lines were computed from the theoretical considerations made above. The bifurcation diagram discussed so far suggests that at the critical points $\eta = \pm 1$ two stable branches merge with an unstable branch. Consequently, we have pitchfork bifurcations. At $\eta = 1$ a single stable branch for $\eta > 1$ emerges in an ordinary pitchfork bifurcation. In contrast, at $\eta = -1$ a single stable branch for $\eta < -1$ emerges in an inverse pitchfork bifurcation. These branches are shown in Fig. 2 for $\eta > 1$ and $\eta < -1$ as well. The circles shown in Fig. 2 were obtained from numerical simulations (see below).
Multifrequency synchronization in canonical dissipative systems

Fig. 2. Bifurcation diagram displaying the fixed point values $\psi_{st}$ of the relative phase characterizing the limit cycle attractors of the coupled CDO model defined by Eq. (21). For $\eta \in [-1, 1]$ the values were computed from Eq. (30). The two side branches for $|\eta| > 1$ are predicted from dynamical systems theory (see text) and reflect the branches of pitchfork bifurcations. Lower part: relations between pseudo-invariants and their corresponding pumping parameters in the regions $\eta < -1$, $\eta \in [-1, 1]$, and $\eta > 1$.

2.3.5. Numerics and the case $|\eta| > 1$

In order to verify the bifurcation diagram shown in Fig. 2, we solved Eqs. (21) numerically using a standard Euler forward method. We used the parameters $m = 2\text{kg}$, $K = 3\text{N/m}$, $\gamma_1 = \gamma_2 = 0.1\text{kg/(Js)}$, $\gamma_3 = 0.05\ 1/(\text{m}^2\text{s})$, $B_1 = 1\text{J}$, and $B_2 = 2\text{J}$. Consequently, we had an oscillation frequency $\omega \approx 1.22\text{rad/s}$ or $f \approx 0.2\text{Hz}$ (an oscillation period of about $5\text{s}$) and a critical value $B_{3,\text{crit}} \approx 2.3\text{kgm}^2/\text{s}$. We varied $B_3$ in the range from $-4, \ldots, 4\text{kgm}^2/\text{s}$ in steps of $1$ unit. In doing so, we obtained a set of bifurcation parameter values $\eta$ in the range of $-1.7, \ldots, +1.7$. Figure 3 shows details of a simulation with $B_3 = 1\text{kgm}^2/\text{s}$, which corresponds to $\eta \approx 0.43$. Panel A shows $q_1$ and $q_2$ as functions of time for a total simulation duration of $200\text{s}$. After a transient period, the coupled oscillators approached a limit cycle attractor. Panel B shows the last $20\text{s}$ out of the $200\text{s}$ period (with $q_1$ given by the solid line and $q_2$ by the dashed line). From the peak to peak intervals of the variables $q_1$ and $q_2$ we determined numerically the relative phase $\psi$. We obtained a value of $\psi \approx 25.74^\circ$, which is consistent with the theoretical value of $\psi_{st} = 25.66^\circ$ as obtained from Eq. (30). Panel C shows the Hamiltonian functions $H_1$ and $H_2$ as functions of time over the total simulation period of $200\text{s}$. The dotted lines correspond to $B_1$ and $B_2$. As expected, the pseudo-invariants converged to the pumping parameters $B_1$ and $B_2$. Finally, panel D shows $L_0$ as a function of time. The dotted horizontal line indicates the parameter value $B_3 = 1\text{kgm}^2/\text{s}$. We found that $L_0$ approached the parameter $B_3$ when the dynamics of the coupled CDO model approached the limit cycle attractor.

As mentioned above, for a simulation period of $T = 200\text{s}$ we numerically determined the relative phase $\psi$ from the last $20$ second interval and thus obtained a value of $\psi \approx 25.74^\circ$. In order to ensure that we captured the dynamics on the limit cycle attractor, we repeated the simulation for a longer period of $T = 600\text{s}$ and again determined the relative difference $\psi$. Within the accuracy of the numerical simulation scheme, the value such obtained did not differ from the previously obtained (i.e., we again obtained a value of $\psi \approx 25.74^\circ$). This value was plotted in the bifurcation diagram shown in Fig. 2 as a circle. Likewise, for $B_3 = -4, -3, -2, -1, 0, 1, 2, 3$ and $4\text{kgm}^2/\text{s}$ the stationary relative phase values $\psi$ were obtained for $T = 600\text{s}$ simulations and plotted in Fig. 2 as circles. We varied the initial conditions $q_1(0)$, $q_2(0)$, $p_1(0)$, and $p_2(0)$ appropriately such that the numerical solutions of the CDO dynamics approached all possible branches in the bistable parameter domain $\eta = (-1, 1)$. As can be seen in Fig. 2, the numerically obtained
values for the relative phases $\psi_{st}$ were consistent with the analytical values obtained from Eq. (30).

As mentioned above, the discussion of the bifurcation diagram for $\eta \in (-1, 1)$ suggests that at $\eta = \pm 1$ the relative phase exhibits pitchfork bifurcations such that single stable branches for $\eta > 1$ and $\eta < -1$ emerge. The simulations results confirmed that for $\eta \geq 1$ the CDO model exhibits a limit cycle attractor with $\psi = 90^\circ$, while for $\eta \leq -1$ the model exhibits a limit cycle attractor with $\psi = -90^\circ$ (or $\psi = 270^\circ$). Let us elaborate on the nature of those limit cycle attractors. To this end, we first present simulation results showing the same quantities as Fig. 3. Panel A demonstrates the approach of the position variables to the limit cycle. Panel B illustrates the (theoretically predicted) phase shift of $\psi = 90^\circ$ when the system is on the limit cycle attractor. From our considerations in Sec. 2.3.2 it follows that the equalities $I_k = B_k$ cannot be satisfied simultaneously for $k = 1, 2, 3$ in the case $\eta > 1$. Panel C shows $H_1$ (i.e., $J_1$) and $H_2$ (i.e., $J_2$) as functions of time. They became stationary when the system dynamics approached the attractor. However, they assumed stationary values larger than their correspond pumping parameters $B_1$ and $B_2$. Panel D shows $L_0$ as a function of time. $L_0$ approached a fixed point value smaller than $B_3$ but larger than $B_{3,\text{crit}}$. Let us show that these observations are consistent with the amplitude equations derived in Sec. 2.3.3.

To begin with, for $\psi = \pi/2$ the sine functions in Eq. (33) are equal to 1. Our working hypothesis (supported by the numerical simulation results shown in panel D) is that for $\eta > 1$ the pseudo-invariant $L_0$ assumes a value smaller than $B_3$. Consequently, the terms $X = -\sin(\psi)(L_0 - B_3)$ in the dynamics for the real-valued amplitudes $R_1$ and $R_2$ are positive. This implies that in the evolution equations for $R_1$ and $R_2$ additional pumping terms occur for the case $\eta > 1$. These pumping terms make it such that $R_1$ and $R_2$ become larger than the values $R_{1,0} = \sqrt{B_1/(2K)}$ and $R_{2,0} = \sqrt{B_2/(2K)}$ obtained in Eq. (25) from the assumption $H_1 = B_1$ and $H_2 = B_2$. In short, from $L_0 < B_3$ it follows that $R_1$ and $R_2$ approach stationary values $R_{1,\text{st}}$ and $R_{2,\text{st}}$ with $R_{1,\text{st}} = R_{1,0} + \epsilon_1$ and $R_{2,\text{st}} = R_{2,0} + \epsilon_2$ with $\epsilon_1, \epsilon_2 > 0$. However, since $R_1 = R_{1,0}$ and $R_2 = R_{2,0}$ is equivalent to $H_1 = B_1$ and $H_2 = B_2$, from $R_{1,\text{st}} = R_{1,0} + \epsilon_1$ and $R_{2,\text{st}} = R_{2,0} + \epsilon_2$ with $\epsilon_1, \epsilon_2 > 0$ it then follows that the Hamiltonian functions assume values larger than their corresponding pumping parameters: $H_1 > B_1$ and $H_2 > B_2$. Therefore, the limit cycle attractor for $\eta > 1$ with $\psi_{st} = \pi/2$ is characterized by the three inequalities $H_1 > B_1$, $H_2 > B_2$, and $L_0 < B_3$ as indicated in the lower part of

![Fig. 3](image-url)
Fig. 4. As in Fig. 3, a numerical solution of the coupled CDO model (21) is shown but for the case $B_3 > B_{3,\text{crit}}$ with $B_3 = 3 \text{kgm}^2/\text{s}$.

Fig. 2. As indicated in Fig. 2 we can also argue that the inequality $L_0 > B_{3,\text{crit}}$ is satisfied. The reason for this is that Eq. (24) holds in general (i.e., irrespective of whether or not $L_0 = B_3$). In line with Eq. (25), we define the RHS factor by $\xi = 4\omega m R_1 R_2$, which implies $L_0 = \xi \sin(\psi)$ and $L_0 = \xi$ for $\psi = \pi/2$. If $R_1 = R_{1,0} + c_1$ and $R_2 = R_{2,0} + c_2$ with $c_1, c_2 > 0$, we have an RHS factor larger than $B_{3,\text{crit}}$, which implies $L_0 > B_{3,\text{crit}}$.

3. Multifrequency synchronization

3.1. Preliminary remarks

Consider two oscillators with angular frequencies $\omega_j$, $j = 1, 2$, such that $n\omega_1 = m\omega_2$, where $n, m$ are integers. For the sake of simplicity, we put either $n = 1$ or $m = 1$ such that $\omega_1 = n\omega_2$ or $\omega_2 = n\omega_1$ (however, future work may consider more general cases with $n \neq 1 \land m \neq 1$). Let us consider harmonic oscillators with slowly evolving phases $\phi_j(t)$ and fixed amplitudes $r_j$ such that

$$q_j(t) = r_j \cos(\omega_j t + \phi_j(t))$$

holds. The phases $\phi_j(t)$ occurring in Eq. (43) define phase shifts relative to the oscillation periods $T_j = 2\pi/\omega_j$. The corresponding phase shifts $\Delta T_j$ in time units are given by

$$\Delta T_j(t) = T_j \frac{\phi_j(t)}{2\pi} = \frac{\phi_j(t)}{\omega_j}. \quad (44)$$

Accordingly, the relative phase shift $\Delta T_{rel}$ in time units can be defined by

$$\Delta T_{rel}(t) = \Delta T_1(t) - \Delta T_2(t) = \frac{\phi_1(t)}{\omega_1} - \frac{\phi_2(t)}{\omega_2}. \quad (45)$$

Let us consider the two aforementioned special cases $n > 1, m = 1$ and $n = 1, m > 1$. If $j = 1$ is the slow oscillator with $\omega_2 = n\omega_1 \Rightarrow \omega_1 = \omega_2/n$ and $n > 1$, we obtain

$$\Delta T_{rel}(t) = \frac{n\phi_1(t)}{\omega_2} - \frac{\phi_2(t)}{\omega_2} = \frac{n\phi_1(t) - \phi_2(t)}{\omega_2} = \frac{1}{\omega_2} \psi(t) \quad (46)$$

with the relative phase

$$\psi(t) = n\phi_1(t) - \phi_2(t) \quad (47)$$
In contrast, if $j = 2$ is the slow oscillator with $\omega_1 = m\omega_2 \Rightarrow \omega_2 = \omega_1/m$ and $m > 1$, we obtain
\begin{equation}
\Delta T_{rel}(t) = \frac{\dot{\phi}_1 - m\dot{\phi}_2(t)}{\omega_1} = \frac{1}{\omega_1} \psi(t), \psi(t) = \phi_1(t) - m\phi_2(t).
\end{equation}

The two oscillator system exhibits multifrequency synchronization if, from a dynamical systems perspective, the relative phase shift $\Delta T_{rel}(t)$ is a dynamical process with at least one asymptotically stable fixed point. This implies that the oscillators show multifrequency synchronization if the relative phase $\psi$ (defined by $\psi(t) = n\phi_1(t) - \phi_2(t)$ for $\omega_1 < \omega_2$ or $\psi(t) = \phi_1(t) - m\phi_2(t)$ for $\omega_2 < \omega_1$) satisfies a dynamical process that exhibits at least one asymptotically stable fixed point. In other words, the oscillator system exhibits at least one limit cycle attractor that is characterized by a relative phase $\psi$ with an asymptotically stable fixed point $\psi_{st}$.

### 3.2. Holt CDOs and 1:2 synchronization

#### 3.2.1. Model

The Holt oscillator is based on the Hamiltonian functions (see Eq. (175) in Holt [1982], see also Bonatsos et al. [1994])
\begin{equation}
H_1 = \frac{p_1^2}{2m} + \frac{Kq_1^2}{2}, \quad H_2 = \frac{p_2^2}{2m} + 2Kq_2^2.
\end{equation}

The harmonic oscillators defined by Eqs. (6) and (49) exhibit oscillation frequencies of $\omega_1 = \sqrt{K/m}$ and $\omega_2 = 2\omega_1 = 2\sqrt{K/m}$. It has been shown that the oscillators exhibit a third invariant (see Eq. (176) in Holt [1982], see also Bonatsos et al. [1994]) defined by
\begin{equation}
I_3 = p_1 \left(\frac{p_1p_2}{m} + 3Kxy\right) - KxL_0,
\end{equation}
where $L_0$ is the (ordinary) angular momentum defined by Eq. (19). In line with our general considerations in Sec. 2.2, the uncoupled Holt CDOs satisfy Eq. (9) with the Hamiltonian functions given by Eq. (49). In the uncoupled case given by Eq. (9) for $B_1, B_2 > 0$ the dynamics of the oscillators converges to limit cycle attractors with $q_j(t) = A_{j,\text{st}} \exp\{i\omega_j t\} + \text{c.c.}$ and stationary complex-valued amplitudes $A_{j,\text{st}}$ with $|A_{j,\text{st}}| > 0$. From the property that $I_3$ is an invariant of the dynamics (6), it follows that on the limit cycle attractors of the uncoupled CDOs (9) the function $I_3$ is an invariant again. However, during transient periods, $I_3$ (just like $H_1$ and $H_2$) varies over time. The coupled Holt CDOs are defined by Eq. (11) with $I_1 = H_1$ and $I_2 = H_2$ given by Eq. (49) and $I_3$ defined by Eq. (50). Equation (11) in this case reads explicitly
\begin{align*}
\frac{d}{dt}p_1 &= -Kq_1 - \gamma_1 \frac{p_1}{m}(H_1 - B_1) - \gamma_3 \left(\frac{2p_1p_2}{m} + 4Kq_1q_2\right)(I_3 - B_3), \quad \frac{d}{dt}q_1 = \frac{p_1}{m},
\frac{d}{dt}p_2 &= -4Kq_2 - \gamma_2 \frac{p_2}{m}(H_2 - B_2) - \gamma_3 \left(\frac{p_1^2}{m} - Kq_1^2\right)(I_3 - B_3), \quad \frac{d}{dt}q_2 = \frac{p_2}{m}.
\end{align*}

The coupled Holt CDOs exhibit the functions $H_1, H_2, I_3$ as invariants in the special case when $G_{tot}$ becomes stationary such that the individual terms $G_j$ with $j = 1, 2, 3$ become stationary as well. In general, the functions $H_1, H_2, I_3$ vary over time. Therefore, they are considered as pseudo-invariants of the model.

#### 3.2.2. Pseudo-invariants and 1:2 multifrequency-synchronization limit cycle attractors

Using Eqs. (14) and (22) the pseudo-invariants of the coupled Holt CDOs can be expressed as
\begin{align}
I_1 &= 2K|A_1|^2 = 2KR_1^2, \quad I_2 = 8K|A_2|^2 = 8KR_2^2,
\end{align}
and
\begin{align}
I_3 &= 16Km\omega_1 \Re(iA_1^2A_2^2) = 16Km\omega_1 R_1^2 R_2 \sin(\phi_2 - 2\phi_1).
\end{align}
Let us define the relative phase
\[ \psi = \phi_2 - 2\phi_1 \] (54)
as introduced in Sec. 3.1. Then, \( I_3 \) depends only on \( \psi \) rather than on \( \phi_1 \) and \( \phi_2 \). From Eq. (51) it follows that there exist limit cycle attractors with \( G_{tot} = 0 \) and \( I_k = B_k \) for \( k = 1, 2, 3 \). These attractors are described by
\[ I_1 = B_1 \Rightarrow R_1 = \sqrt{B_1/(2K)} \ , \quad I_2 = B_2 \Rightarrow R_2 = \sqrt{B_2/(8K)} \ , \]
\[ I_3 = B_3 \Rightarrow R_3 = \frac{16K \omega_1 R_1^2 R_1 \sin(\psi_{st})}{\xi} \] (55)

Accordingly, let us define \( B_{3,\text{crit}} \) as the RHS factor \( \xi' \) for which the conditions \( I_1 = B_1 \) and \( I_2 = B_2 \) are satisfied:
\[ B_{3,\text{crit}} = 16mK\omega_1 R_1^2 (B_1) R_1 (B_2) = 2B_1 \sqrt{2mB_2} \] (56)

By definition, limit cycle attractors with \( I_k = B_k \) and \( k = 1, 2, 3 \) exist only when all three conditions
\[ R_1 = \sqrt{B_1/(2K)} \ , \quad R_2 = \sqrt{B_2/(8K)} \ , \quad B_3 = B_{3,\text{crit}} \sin(\psi_{st}) \] (57)
are satisfied. By analogy to our considerations in Sec. 2.3, the three conditions in Eq. (57) can be satisfied simultaneously only if \( B_3 \in [-B_{3,\text{crit}}, B_{3,\text{crit}}] \) or \( \eta \in [-1, 1] \) holds.

3.2.3. Amplitude equations

The coupled Holt CDO model (51) can be cast into the second-order Newtonian oscillator equations
\[
\frac{d^2}{dt^2} q_1 = -\frac{K}{m} q_1 - \gamma_1 \frac{p_1}{m^2} (H_1(A_1) - B_1) - \frac{\gamma_3}{m} \left( \frac{2p_1 p_2}{m} + 4k q_1 q_2 \right) (I_3(A_1, A_2) - B_3) ,
\]
\[
\frac{d^2}{dt^2} q_2 = -\frac{4K}{m} q_2 - \gamma_2 \frac{p_2}{m^2} (H_2(A_2) - B_2) - \frac{\gamma_3}{m} \left( \frac{p_1^2}{m} - K q_1^2 \right) (I_3(A_1, A_2) - B_3) .
\] (58)

Substituting Eqs. (14) and (22) into Eq. (58), we obtain the amplitude equations
\[
\frac{d}{dt} A_1 = \frac{\gamma_1}{2m} A_1 (H_1(A_1) - B_1) + 4i \gamma_3 \omega_1 A_1^* A_2 (I_3(A_1, A_2) - B_3) ,
\]
\[
\frac{d}{dt} A_2 = \frac{\gamma_2}{2m} A_2 (H_2(A_2) - B_2) - i \frac{\gamma_3 \omega_1}{2} A_1^* (I_3(A_1, A_2) - B_3) .
\] (59)

In polar coordinates, Eq. (59) reads
\[
\frac{d}{dt} R_1 = \frac{\gamma_1}{2m} R_1 (H_1(R_1) - B_1) - 4\gamma_3 \omega_1 R_1 R_2 \sin(\psi) (I_3(R_1, R_2, \psi) - B_3) ,
\]
\[
\frac{d}{dt} R_2 = \frac{\gamma_2}{2m} R_2 (H_2(R_2) - B_2) - \frac{\gamma_3 \omega_1}{2} R_1^2 \sin(\psi) (I_3(R_1, R_2, \psi) - B_3)
\] (60)

and for \( R_2 > 0 \) we have
\[
\frac{d}{dt} \phi_1 = 4\gamma_3 \omega_1 R_2 \cos(\psi) (I_3(R_1, R_2, \psi) - B_3) ,
\]
\[
\frac{d}{dt} \phi_2 = -\frac{\gamma_3 \omega_1}{2} R_1^2 \cos(\psi) (I_3(R_1, R_2, \psi) - B_3) ,
\] (61)

where \( I_3 \) is defined by Eq. (53) with \( \psi = \phi_2 - 2\phi_1 \), see Eq. (54). Accordingly, the evolution equation for \( \psi \) reads
\[
\frac{d}{dt} \psi = -\frac{\gamma_3 \omega_1}{2R_2} \left( R_1^2 + (4R_2)^2 \right) \cos(\psi) (I_3(R_1, R_2, \psi) - B_3) ,
\]
\[
= -\frac{\gamma_3 \omega_1}{2R_2} \left( R_1^2 + 16R_2^2 \right) \cos(\psi) (I_3(R_1, R_2, \psi) - B_3) .
\] (62)
Equations (60) and (62) correspond to a closed set of differential equations for $R_1$, $R_2$, and $\psi$.

### 3.2.4. Bifurcation diagram and simulations

Equation (62) for $\psi$ has the same general structure

$$\frac{d}{dt}\psi = \gamma_3 \Gamma(R_1, R_2) \cos(\psi)(I_3 - B_3)$$

as Eq. (35). Likewise, Eq. (60) for $R_1$ and $R_2$ has the same structure as Eq. (33). Therefore the arguments made in Sec. 2.3.4 in the case of the coupled CDO model for monofrequency synchronization also apply to the multifrequency case of the coupled Holt CDOs. Accordingly, Fig. 5 summarizes the bifurcation diagram. For $B_3 \in [-B_{3,\text{crit}}, B_{3,\text{crit}}]$ (i.e., $\eta \in [-1, 1]$) there exist two limit cycle attractors with

$q_1 = r_1 \cos(h), \quad q_2 = r_2 \cos(2h + \psi_{st}), \quad \eta = \sin(\psi_{st})$

and $r_1 = 2R_1 = \sqrt{2B_1/K}$, $r_2 = 2R_2 = \sqrt{B_2/(2K)}$, where $h$ is an arbitrary phase shift (as in Eq. (37)). Moreover, for $\eta \in (-1, 1)$ there exist two unstable closed orbits with

$q_1 = r_1 \cos(h), \quad q_2 = r_2 \cos(2h + \psi_{st}), \quad \psi_{st} = \pi/2, 3\pi/2$

and $r_1, r_2$ as for the limit cycle attractors. At $\eta = \pm 1$ (i.e., $B_3 = \pm B_{3,\text{crit}}$) the unstable orbits merge with the limit cycle attractors. With respect to the dynamics of $\psi$, at $\eta = \pm 1$ we have pitchfork bifurcations. For $\eta < -1$ and $\eta > 1$ (i.e., $B_3 < -B_{3,\text{crit}}$ and $B_3 > B_{3,\text{crit}}$) limit cycle attractors with

$q_1 = r_1' \cos(h), \quad q_2 = r_2' \cos(2h + \psi_{st})$

exist with $\psi_{st} = \pi/2$ for $\eta > 1$ and $\psi_{st} = -\pi/2$ (i.e., $\psi_{st} = 3\pi/2$) for $\eta < -1$. The amplitudes $r'_{1,2}$ of those limit cycle attractors may be determined by numerical simulations (see below).

![Bifurcation diagram](image_url)

Fig. 5. Bifurcation diagram based on the relative phase $\psi_{st}$ characterizing 1:2 multifrequency-synchronization limit cycle attractors (see Eqs. (64) and (66)) of the coupled Holt CDO model defined by Eq. (51). The solid lines were computed as in Fig. 2 and in fact are equivalent to those shown in Fig. 2. Squares represent numerically obtained values (see text).

We conducted several simulations in order to demonstrate the bifurcation diagram shown in Fig. 5. The coupled Holt CDOs (51) were solved by a standard Euler forward method for the following parameters:

$m = 2\text{kg}, \quad K = 3\text{N/m}, \quad \gamma_1 = 0.1\text{kg/(Js)}, \quad \gamma_2 = 0.2\text{kg/(Js)}$, $\gamma_3 = 0.02\text{s/(Nkg m^3)}, \quad B_1 = 1\text{J}, \quad \text{and} \quad B_2 = 2\text{J}$.

Consequently, we considered a model with the oscillation frequencies $\omega_1 \approx 1.22\text{rad/s}$ (i.e., $f \approx 0.2\text{Hz}$, which
corresponds to an oscillation period of about 5s) and $\omega_2 \approx 2.45\text{rad/s}$ (i.e., $f \approx 0.4\text{Hz}$, which corresponds to a period of about 2.5s). The model exhibited a critical parameter $B_{3,\text{crit}} \approx 5.7\text{Nkgm}^2/\text{s}$. We varied $B_3$ in the range from $-8, \ldots, 8\text{Nkgm}^2/\text{s}$ in steps of 2 units (which means that $\eta$ was varied from $-1.4$ to 1.4). The CDO model was solved for $T = 300\text{s}$. Visual inspection of trajectories $q_1$ and $q_2$ as well as pseudo-invariants showed that at $T = 300\text{s}$ the oscillator system evolved on one of its limit cycle attractors (see below for two illustrative examples). The phases $\phi_1$ and $\phi_2$ were determined numerically from the last 30-second period of the $T = 300\text{s}$ trajectories with respect to the reference time point $t = 270\text{s}$. From the individual phases the relative phase $\psi_{st}$ was computed. The relative phases thus obtained for the parameters $B_3 = -8, -6, -4, -2, 0, 2, 4, 6, 8$ are shown in Fig. 5 as squares. The initial conditions $q_1(0), q_2(0), p_1(0), p_2(0)$ were varied such that the oscillator system settled down to all branches of the bifurcation diagram. As can be seen in Fig. 5, the numerically obtained values for the relative phases $\psi_{st}$ were consistent with the analytical values obtained from Eq. (30).

Fig. 6. A solution of the coupled Holt CDO system (51) obtained by solving Eq. (51) numerically for $B_3 = 2\text{Nkgm}^2/\text{s}$ exemplifying the parameter domain $B_3 < B_{3,\text{crit}}$. Panel A: $q_1$ (top) and $q_2$ (bottom). Panel B: Detail of panel A from $t = 90\text{s}$ to $t = 100\text{s}$ (for the sake of simplicity the horizontal axis has been relabeled) with $q_1$ and $q_2$ given by the solid and dashed lines, respectively. Panel C: $H_1$ and $H_2$ (the dotted lines correspond to $B_1$ and $B_2$). Panel D: $I_3$ (the dotted line corresponds to $B_3$).

Figures 6 and 7 show details from two simulations. In those trials only trajectories for $T = 100\text{s}$ were simulated. Figure 6 shows simulation results for $B_3 = 2\text{Nkgm}^2/\text{s}$ (i.e., $\eta \approx 0.35$) for initial conditions such that the CDO system converged to an attractor with fixed point $\psi_{st}$ on the lower branch $\psi_{st} \in (0, \pi/2)$ in Fig. 5. Using the last 10s to determine $\psi_{st}$, we obtained $\psi_{st} \approx 20.84^\circ$ (the analytical value from Eq. (30) is $\psi_{st} = 20.70^\circ$; in fact, for the longer simulation with $T = 300\text{s}$ reported in Fig. 5 we obtained $\psi_{st} \approx 20.61^\circ$, which is somewhat closer to the analytical value). Let us return to Fig. 6. Panel A shows $q_1$ and $q_2$ as functions of time. The graphs demonstrate the approach towards a limit cycle attractor. Panel B shows $q_1$ (solid line) and $q_2$ (dashed line) for the last 10s period (from $t = 90\text{s}$ to $t = 100\text{s}$) shown in panel A. It is clear that $q_2$ exhibited oscillations with twice the oscillation frequency as $q_1$. Panel C displays the pseudo-invariants $H_1$ and $H_2$ during the $T = 100\text{s}$ period. As expected, the pseudo-invariants approached the parameter values $B_1 = 1$ and $B_2 = 2$, respectively. Panel D shows the coupling function $I_3$ over time. Just like $H_1$ and $H_2$, we see that $I_3$ converged to $B_3$, as predicted.

While Fig. 6 exemplifies the coupled CDO dynamics for the domain $|\eta| \leq 1$, Fig. 7 illustrates the situation in the parameter domain $|\eta| > 1$. Figure 7 shows simulation results for $B_3 = 8\text{Nkgm}^2/\text{s}$ (i.e.,
$\eta \approx 1.41$). Just as in Fig. 6, panels A, B, C and D of Fig. 7 show $q_1$ and $q_2$ for a simulation period of $T = 100s$ (panel A) and the last 10s of that period (panel B), as well as $H_1$, $H_2$ (panel C) and $I_3$ (panel D) for the full 100s period. Unlike Fig. 6, the pseudo-invariants in Fig. 7 do not approach the parameter values $B_1$, $B_2$, $B_3$. In line with our discussion in Sec. 2.3.4 on the coupled CDOs for the monofrequency case, we see that $I_3$ assumed a value smaller than $B_3$ (but larger than $B_{3,crit}$), whereas $H_1$ and $H_2$ assumed values larger than $B_1$ and $B_2$, respectively. This general property of the pseudo-invariants for $|\eta| > 1$ is also indicated in the bottom part of Fig. 5.

### 3.3. Fokas-Lagerstrom CDOs and 1:3 synchronization

#### 3.3.1. Model

Fokas and Lagerstrom examined two harmonic oscillators defined by Eq. (6) and the Hamiltonian functions (see Eq. (3.15b) in Fokas & Lagerstrom [1980], see also Bonatsos et al. [1994])

$$H_1 = \frac{p^2_1}{2m} + K\frac{q^2_1}{2}, \quad H_2 = \frac{p^2_2}{2m} + K\frac{q^2_2}{18}. \quad (67)$$

Accordingly, the oscillators exhibit angular oscillation frequencies $\omega_1 = \sqrt{K/m}$ and $\omega_2 = \omega_1/3 = 3^{-1}\sqrt{K/m}$. Here, oscillator 1 is the fast oscillator and oscillator 2 is the slow oscillator. Among other things, Fokas & Lagerstrom [1980] showed that the dynamics of the two oscillators exhibits a third invariant defined by

$$I_3 = \frac{L_0}{m}p^2_2 - \frac{K}{3}L_{adj}q^2_2, \quad (68)$$

where $L_0$ is the angular momentum defined by Eq. (19) and $L_{adj}$ is an "adjusted" angular momentum given by

$$L_{adj} = q_1p_2 - \frac{1}{9}q_2p_1. \quad (69)$$

Let us define the uncoupled Fokas-Lagerstrom CDOs by Eqs. (9) and (67). For $B_1, B_2 > 0$ the dynamics of the uncoupled CDOs converges to limit cycle attractors with $q_j(t) = A_{j, st} \exp\{i\omega_j t\} + c.c.$ and stationary complex-valued amplitude $A_{j, st}$ with $|A_{j, st}| > 0$. As argued in Sec. 2.2 the variable $I_3$ becomes an invariant
of the uncoupled Fokas-Lagerstrom CDO system (9) in the long time limit when the system settles down on such a limit cycle attractor. The coupled Fokas-Lagerstrom CDO system is defined by Eq. (11) with \( I_1 = H_1, I_2 = H_2, \) and \( I_3 \) defined by Eqs. (67) and (68). Accordingly, the coupled Fokas-Lagerstrom CDO model explicitly reads

\[
\begin{align*}
\frac{d}{dt} p_1 &= -K q_1 - \gamma_1 \frac{p_1}{m} (H_1 - B_1) - \gamma_3 q_2 \left( \frac{K}{27} q_2^2 - \frac{p_2^3}{m} \right) (I_3 - B_3), \\
\frac{d}{dt} p_2 &= -\frac{K}{9} q_2 - \gamma_2 \frac{p_2}{m} (H_2 - B_2) - \gamma_3 \left( \frac{p_2^2}{m} [3q_1 p_2 - 2 q_2 p_1] - \frac{K}{3} q_1 q_2^2 \right) (I_3 - B_3),
\end{align*}
\]

\( I_3 = \frac{16}{27} K m \omega_1 \text{Re}(i A_1 A_2^*)^3 = \frac{16}{27} K m \omega_1 R_1 R_2^3 \sin(3\phi_2 - \phi_1). \)

The variables \( I_1 = H_1, I_2 = H_2, \) and \( I_3 \) are pseudo-invariants of the dynamics defined by the model (70) in the sense discussed in Sec. 2.2.

### 3.3.2. Pseudo-invariants and 1:3 multifrequency-synchronization limit cycle attractors

Using Eqs. (14) and (22), the invariants of the coupled Fokas-Lagerstrom CDOs read

\[
I_1 = 2K|A_1|^2 = 2KR_1^2, \quad I_2 = \frac{2}{9} K |A_2|^2 = \frac{2}{9} KR_2^2,
\]

and

\[
I_3 = \frac{16}{27} K m \omega_1 \text{Re}(i A_1 A_2^*)^3 = \frac{16}{27} K m \omega_1 R_1 R_2^3 \sin(3\phi_2 - \phi_1).
\]

Let us define for the Fokas-Lagerstrom model the relative phase

\[
\psi = 3\phi_2 - \phi_1
\]

as introduced in Sec. 3.1 such that \( I_3 \) depends only on \( \psi \). From Eq. (70) and our general discussion in Sec. 2.2 it follows that there exist limit cycle attractors with \( G_{\text{tot}} = 0 \) and \( I_k = B_k \) with \( k = 1, 2, 3 \) characterized by

\[
I_1 = B_1 \Rightarrow R_1 = \sqrt{B_1/(2K)}, \quad I_2 = B_2 \Rightarrow R_2 = 3\sqrt{B_2/(2K)}, \\
I_3 = B_3 \Rightarrow R_3 = \frac{16}{27} K m \omega_1 \text{Re}(i A_1 A_2^*)^3 \sin(\psi_{\text{st}}).
\]

As in Sections 2.3 and 3.2, we define \( B_{3,\text{crit}} \) as the RHS factor \( \xi'' \) for which the conditions \( I_1 = B_1 \) and \( I_2 = B_2 \) are satisfied:

\[
B_{3,\text{crit}} = \frac{16}{27} K m \omega_1 R_1 (B_1) R_2^3 (B_2) = \frac{4}{\omega_1} \sqrt{B_1 B_2^3}
\]

Consequently, limit cycle attractors with \( I_k = B_k \) and \( k = 1, 2, 3 \) are characterized by

\[
R_1 = \sqrt{B_1/(2K)}, \quad R_2 = 3\sqrt{B_2/(2K)}, \quad B_3 = B_{3,\text{crit}} \sin(\psi_{\text{st}})
\]

and exist only when those three conditions are simultaneously satisfied. Introducing once more the bifurcation parameter \( \eta = B_3/B_{3,\text{crit}} \), Eq. (76c) becomes Eq. (30). Just as we saw in the cases in Sections 2.3 and 3.2, the conditions in Eq. (76) can be satisfied only for \( B_3 \in [-B_{3,\text{crit}}, B_{3,\text{crit}}] \) or \( \eta \in [-1, 1] \).

### 3.3.3. Amplitude equations

The Fokas-Lagerstrom model (70) can be equivalently expressed as

\[
\begin{align*}
\frac{d^2}{dt^2} q_1 &= -K q_1 - \gamma_1 \frac{p_1}{m^2} (H_1(A_1) - B_1) - \gamma_3 q_3 \left( \frac{K}{27} q_2^2 - \frac{p_2^3}{m} \right) (I_3 - B_3), \\
\frac{d^2}{dt^2} q_2 &= -\frac{K}{9m} q_2 - \gamma_2 \frac{p_2}{m^2} (H_2(A_2) - B_2) - \gamma_3 \left( \frac{p_2^2}{m} [3q_1 p_2 - 2 q_2 p_1] - \frac{K}{3} q_1 q_2^2 \right) (I_3 - B_3).
\end{align*}
\]
Substituting Eqs. (14) and (22) into Eq. (77) and using the rotating wave and slowly varying amplitude approximation methods [Haken, 1985] once more, we obtain

\[\begin{align*}
\frac{d}{dt} A_1 &= \frac{\gamma_1}{2m} A_1 (H_1(A_1) - B_1) + \frac{2}{27} i \gamma_3 \omega_1 A_3^3 (I_3(A_1, A_2) - B_3), \\
\frac{d}{dt} A_2 &= \frac{\gamma_2}{2m} A_2 (H_2(A_2) - B_2) - 2 i \gamma_3 \omega_1 A_1 (A_2^2) (I_3(A_1, A_2) - B_3).
\end{align*}\]  

(78)

In polar coordinates Eq. (78) reads

\[\begin{align*}
\frac{d}{dt} R_1 &= \frac{\gamma_1}{2m} R_1 (H_1(R_1) - B_1) - \frac{2}{27} \gamma_3 \omega_1 R_3^3 \sin(\psi) (I_3(R_1, R_2, \psi) - B_3), \\
\frac{d}{dt} R_2 &= \frac{\gamma_2}{2m} R_2 (H_2(R_2) - B_2) - 2 \gamma_3 \omega_1 R_1 R_2^2 \sin(\psi) (I_3(R_1, R_2, \psi) - B_3)
\end{align*}\]  

(79)

and for \( R_1 > 0 \) we get

\[\begin{align*}
\frac{d}{dt} \phi_1 &= \frac{2}{27} \gamma_3 \omega_1 \frac{R_3^3}{R_1} \cos(\psi) (I_3(R_1, R_2, \psi) - B_3), \\
\frac{d}{dt} \phi_2 &= -2 \gamma_3 \omega_1 R_1 R_2 \cos(\psi) (I_3(R_1, R_2, \psi) - B_3).
\end{align*}\]  

(80)

From Eq. (80) it follows that the evolution equation for the relative phase \( \psi = 3\phi_2 - \phi_1 \) is given by

\[\begin{align*}
\frac{d}{dt} \psi &= -\frac{2 \gamma_3 \omega_1 R_2}{27 R_1} ((9R_1)^2 + R_2^2) \cos(\psi) (I_3(R_1, R_2, \psi) - B_3) \\
&= -\frac{2 \gamma_3 \omega_1 R_2}{27 R_1} ((81R_1^2 + R_2^2) \cos(\psi) (I_3(R_1, R_2, \psi) - B_3)
\end{align*}\]  

(81)

Eqs. (79) and (81) are a closed set of differential equations for \( R_1, R_2, \) and \( \psi. \)

### 3.3.4. Bifurcation diagram and simulations

Just as in the case of the coupled Holt CDO model, the evolution equation (81) for \( \psi \) has the same general structure (63) as Eq. (35). Likewise, the evolution equations (79) for the amplitudes \( R_1 \) and \( R_2 \) have the same structure as Eq. (33). Therefore the arguments made in Sec. 2.3.4 for the case of the coupled CDO model for monofrequency synchronization also apply to the multifrequency case of the coupled Fokas-Lagerstrom CDOs. Consequently, the bifurcation diagram corresponds to the previously discussed diagrams of Figs. 2 and 5. For the sake of completeness and in order to present the results of numerical simulations, the bifurcation diagram of the coupled Fokas-Lagerstrom CDO model is presented explicitly as Fig. 8. For \( B_3 \in [-B_3, crit, B_3, crit] \) or \( \eta \in [-1, 1] \) two limit cycle attractors with

\[\begin{align*}
q_1 &= r_1 \cos(3h - \psi_{st}), \\
q_2 &= r_2 \cos(h), \\
\eta &= \sin(\psi_{st}),
\end{align*}\]  

(82)

exist for \( r_1 = 2R_1 = 2\sqrt{B_1/(2K)} \) and \( r_2 = 2R_2 = 6\sqrt{B_2/(2K)} \), where \( h \) is an arbitrary phase shift. For \( \eta \in (-1, 1) \) there also exist two unstable closed orbits with

\[\begin{align*}
q_1 &= r_1 \cos(3h - \psi_{st}), \\
q_2 &= r_2 \cos(h), \\
\psi_{st} &= \pi/2, 3\pi/2
\end{align*}\]  

(83)

and \( r_1, r_2 \) as for the limit cycle attractors. At \( \eta = \pm 1 \) (i.e., \( B_3 = \pm B_3, crit \)) we observe pitchfork bifurcations such that for \( \eta < -1 \) and \( \eta > 1 \) (i.e., \( B_3 < -B_3, crit \) and \( B_3 > B_3, crit \)) limit cycle attractors with

\[\begin{align*}
q_1 &= r_1' \cos(3h - \psi_{st}), \\
q_2 &= r_2' \cos(h)
\end{align*}\]  

(84)

exist with \( \psi_{st} = \pi/2 \) for \( \eta > 1 \) and \( \psi_{st} = -\pi/2 \) (i.e., \( \psi_{st} = 3\pi/2 \)) for \( \eta < -1 \) and amplitudes \( r_{1,2}' \) that may be determined by numerical simulations (see below).

The bifurcation diagram shown in Fig. 8 and the existence of the corresponding limit cycle attractors was verified by numerical simulations of the coupled Fokas-Lagerstrom CDOs (70) using a standard Euler forward method. The following parameters were used: \( m = 2 \text{kg}, \ K = 3 \text{N/m}, \ \gamma_1 = \gamma_2 = 0.1 \text{kg/(Js)}, \ \gamma_3 = 0.001, \ B_1 = 1 \text{J}, \) and \( B_2 = 2 \text{J}. \) Accordingly, we simulated two coupled CDO oscillators with \( \omega_1 \approx 1.22 \text{rad/s} \)
Fig. 8. Bifurcation diagram describing 1:3 multifrequency-synchronization limit cycle attractors of the coupled Fokas-Lagerstrom CDO model (70) in terms of the relative phase $\psi_{st}$ and Eqs. (82) and (84). The solid lines were computed as in Fig. 2 and in fact are equivalent to those shown in Fig. 2. Diamonds represent numerically obtained values (see text).

(i.e., $f \approx 0.2$Hz and a period of about 5s) and $\omega_2 \approx 0.41$rad/s (i.e., $f \approx 0.07$Hz and a period of about 15s). The model exhibited a critical bifurcation parameter $B_{3, crit} \approx 9.2$ kg$^2$m$^4$/s$^3$. The parameter $B_3$ was varied in the range from $-16,...,16$ kg$^2$m$^4$/s$^3$ in steps of 4 units (which means that $\eta$ was varied from −1.7 to 1.7). The CDO model (70) was solved for a duration $T = 600s$ that was for all parameter values $B_3 \in [-16,16]$ sufficiently long such that (by visual inspection) the trajectories $q_1(t), p_1(t)$ and $q_2(t), p_2(t)$ approached limit cycle attractors. The phases $\phi_1$ and $\phi_2$ were determined numerically from the last 60-second period of the $T = 600s$ trajectories with respect to the reference time point $t = 540s$. From the phases $\phi_1$ and $\phi_2$ the relative phase $\psi_{st}$ was computed. The relative phases thus obtained for $B_3 = -16,-12,-8,-4,0,4,8,12,16$ are shown in Fig. 8 as diamonds. Appropriate initial conditions $q_1(0), q_2(0), p_1(0), p_2(0)$ were selected such that the coupled oscillator system settled down to all branches of the bifurcation diagram shown in Fig. 8. We found that the numerically obtained values for the relative phases $\psi_{st}$ were consistent with the analytical values obtained from Eq. (30).

Figures 9 and 10 exemplify the approach of the CDO system to limit cycle attractors for the two qualitatively different parameter domains $|\eta| \leq 1$ (Fig. 9) and $|\eta| > 1$ (Fig. 10). To this end, only trajectories for $T = 300s$ were simulated. $B_3$ was set as $B_3 = 4$ kg$^2$m$^4$/s$^3$ ⇒ $\eta \approx 0.43$ (Fig. 9) and $B_3 = 12$ kg$^2$m$^4$/s$^3$ ⇒ $\eta \approx 1.30$ (Fig. 10), respectively. Panels A and B in Figs. 9 and 10 shows $q_1$ and $q_2$ as functions of time for the whole $T = 300s$ periods (panels A) and the final 30s periods (panels B). The graphs demonstrate that the coupled CDO system approached limit cycle attractors in both cases. Panels C and D show the pseudo-invariants $H_1, H_2$, and $I_3$ during the $T = 300s$ simulation periods. As expected, for $|\eta| \leq 1$ (i.e., Fig. 9) the pseudo-invariants approached their respective parameters $B_k$. In contrast, for $|\eta| > 1$ (i.e., Fig. 10) we found that $H_1 > B_1, H_2 > B_2$ and $B_{3, crit} < I_3 < B_3$.

4. Discussion
We explored the synchronization of two coupled self-oscillators within the framework of canonical-dissipative systems, considering the monofrequency and multifrequency case. The self-oscillators were coupled by means of functions describing third invariants of the oscillator dynamics in the special case when pumping, damping, and coupling can be neglected. In line with the canonical-dissipative approach, those third invariant functions were also invariants of the monofrequency and multifrequency limit cycle attractors when pumping, damping, and coupling were taken into account. Explicitly, we considered three
Fig. 9. A trajectory of the coupled Fokas-Lagerstrom CDO model (70) obtained by solving Eq. (70) numerically for $B_3 = 4kg^2m^4/s^3$. Panel A: $q_1$ (top) and $q_2$ (bottom). Panel B: Detail of panel A from $t = 270s$ to $t = 300s$ (for the sake of simplicity the horizontal axis has been relabeled) with $q_1$ and $q_2$ given by the solid and dashed lines, respectively. Panel C: $H_1$ and $H_2$ (the dotted lines correspond to $B_1$ and $B_2$). Panel D: $I_3$ (the dotted line corresponds to $B_3$).

Fig. 10. As in Fig. 9, a numerical solution of Eq. (70) is shown but for $B_3 = 12kg^2m^4/s^3$.

coupled oscillator systems with 1:1, 1:2, and 1:3 frequency ratios involving the standard CDOs, the Holt CDOs, and the Fokas-Lagerstrom CDOs.

We found that those three systems exhibit the same bifurcation diagram when characterizing the bifurcation diagram with an appropriately defined relative phase $\psi$. For a suitably defined bifurcation parameter $\eta$, we showed that in the domain $|\eta| < 1$ two limit cycle attractors exist that merge via pitchfork bifurcations at $\eta = \pm 1$ such that in the domain $|\eta| > 1$ the systems exhibit only single attractors characterized by relative phases $\psi = \pm \pi$. The attractors for $|\eta| > 1$ can be described semi-analytically. In contrast, for
$|\eta| \geq 1$ exact analytical descriptions of the attractors can be given.

The equivalence of the bifurcations diagrams for all three oscillator systems is due to the fact that the aforementioned third invariant functions are based on the angular momentum of a particle moving in a two-dimensional plane. In particular, the invariant functions are linear with respect to the angular momentum. Likewise, the coupling function $G_3$ used to couple the oscillators was assumed in all cases to be quadratic in the third invariant function. In contrast, a recent study by Mongkolsakulvong & Frank [2017] on coupled CDOs involving Smorodinsky-Winternitz potentials obtained the bifurcation diagram shown in Fig. 11.

Fig. 11. Bifurcation diagram reported in Mongkolsakulvong & Frank [2017] displaying the fixed point values $\psi_{st}$ of the relative phase characterizing the limit cycle attractors of the coupled CDO model defined by Eqs. (11), (18), and (20) with $I_3 = L_0$ replaced by $I_3 = L_2^0/m$. In line with variable replacement $L_0 \rightarrow L_2^0/m$, the parameter $\eta$ is defined by $\eta = B_3/B_3^{*, crit}$ with $B_3^{*, crit} = B_2^{*, crit}/m$ and $B_3^{*, crit}$ defined by Eq. (26), see also Mongkolsakulvong & Frank [2017].

In fact, Fig. 11 refers to the special case when the nonlinear terms of the Smorodinsky-Winternitz potentials were ignored. Consequently, the diagram refers to the CDOs defined by Eqs. (9), (10) and (18) in Sec. 2.3 for the case of monofrequency synchronization. Comparing Fig. 11 with Figs. 2, 5, and 8, we see similarities and differences. Just as we see in Figs. 2, 5, and 8, Fig. 11 describes pitchfork bifurcations at certain critical values for $\eta$ at which limit cycle attractors merge. However, for $\eta > 1$ the coupled CDO system characterized by the bifurcation diagram shown in Fig. 11 still exhibits multiple attractors. Moreover, the lower critical value for $\eta$ is given by 0 rather than by $-1$ and in the domain $\eta < 0$, the system again exhibits multiple attractors. The reason for these differences is that the third invariant function used in the study by Mongkolsakulvong & Frank [2017] was based on the squared angular momentum $L_2^0$, which is indeed the appropriate term in order to discuss the nonlinearities of the Smorodinsky-Winternitz potentials. In the study by Mongkolsakulvong & Frank [2017] the pseudo-invariant $L_2^0/m$ was substituted into the $G_3$ function such that $G_3 = \gamma_3(L_2^0/m - B_3)^2/2$. In other words, the bifurcation diagram shown in Fig. 11 refers to a coupled CDO model defined by Eqs. (11), (18), and (20) with $I_3 = L_0$ replaced by $I_3 = L_2^0/m$. These considerations illustrate that the choice of the $G_3$ coupling function crucially determines what kind of limit cycle attractors can emerge in a coupled CDO system for a given set of model parameters.

As mentioned in the introduction, coupled self-oscillators have been discussed from the perspective of active systems in a series of previous studies [Ebeling et al., 1999; Erdmann & Ebeling, 2005; Erdmann et al., 2000; Schimansky-Geier et al., 2005], see also Fig. 1. These studies have been focused primarily on the Rayleigh oscillator that might be considered to be a more elementary self-oscillator as compared with a canonical-dissipative oscillator. Therefore, the question arises: what is the relationship between CDOs and more elementary oscillators such as the Rayleigh oscillator or the van der Pol oscillator? The answer to the question is twofold. First, CDOs may be considered as oscillators that arise when more elementary
oscillators are combined in a suitable fashion. For example, the standard CDO defined by Eqs. (3) and (4) can be interpreted as a suitable superposition of the Rayleigh and van der Pol oscillator (see, e.g., Dotov & Frank [2011]; Ebeling & Sokolov [2004]; Reit et al. [2014]). In the movement science literature, oscillators arising from combinations of Rayleigh and van der Pol oscillators have been examined in some detail and are referred to as hybrid oscillators [Beek et al., 1995; Kay et al., 1987]. For the electrical engineering sciences this observation may open an avenue to construct electronic realization of CDOs by combining elementary electronic self-oscillators (as mentioned in the introduction). Second, CDOs may be used as approximations to more elementary self-oscillators. The motivation for considering such an approximation is that some analytical results that can be obtained for CDOs [Ebeling et al., 2008; Ebeling & Sokolov, 2004] might be difficult to be derived for other self-oscillators that do not belong to the class of canonical-dissipative systems. For example, in a series of earlier studies [Ebeling & Röpke, 2004; Ebeling et al., 1999; Schimansky-Geier et al., 2005] the fundamental CDO defined by Eqs. (3) and (4) has been used to approximate the Rayleigh oscillator in order to obtain analytical expressions for distribution functions in the case when the oscillator dynamics is perturbed by noise.

Finally, note that in the electrical engineering sciences driven CDOs have been examined [Reit et al., 2014]. While a detailed discussion of this topic is beyond the scope of the present study, there are similarities between two coupled CDOs and a master-slave system, in which the slave is a CDO and the master system is a simple oscillatory driving force. While typically (see e.g. Reit et al. [2014]) the driving force is modeled as an additive force (just like in the case of the ordinary driven pendulum), our considerations in Secs. 2 and 3 suggest that it might be worthwhile to examine master-slave systems composed of CDO oscillators that are coupled to oscillatory driving forces via mechanisms that mimic angular momentums. A benefit of such an engineered coupling might be that under appropriate circumstances the whole coupling mechanism reflects an invariant of the master-slave system, just as in the case of the coupled CDOs systems discussed in the current study.

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